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Classical solutions for Poisson Sigma Models on a Riemann surface

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Abstract: We determine the moduli space of classical solutions to the field equations of Poisson Sigma Models on arbitrary Riemann surfaces for Poisson structures with vanishing Poisson form class. This condition ensures the existence of a presymplectic form on the target Poisson manifold which agrees with the induced symplectic forms of the Poisson tensor upon pullback to the leaves. The dimension of the classical moduli space as a function of the genus of the worldsheet Σ and the corank k of the Poisson tensor is determined as $k\,(\mathrm{rank}(H^1(\Sigma))+1)$. Representatives of the classical solutions are provided using the above mentioned presymplectic 2-forms, and possible generalizations to cases where such a form does not exist are discussed. The results are compared to the known moduli space of classical solutions for two-dimensional BF and Yang-Mills theories.

Keywords: Differential and Algebraic Geometry, Integrable Field Theories, Bosonic Strings.

Contents

1. Introduction

Poisson Sigma Models (PSMs) [1, 2] are topological or almost topological two-dimensional field theories associated to a Poisson manifold. Given any Poisson bracket on a manifold M, characterized by a Poisson bivector $\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(X)\partial_i \wedge \partial_j$ where X^i, $i = 1, \ldots, n$, are local coordinates on M, the topological part of the action has the form

$$S = \int_\Sigma A_i \wedge \mathrm{d}X^i + \frac{1}{2}\mathcal{P}^{ij} A_i \wedge A_j \,. \qquad (1.1)$$

It is a functional of the fields $X(x)$, which parametrize a map $\mathcal{X}$ from the two-dimensional worldsheet Σ into the target M, as well as 1-forms on the worldsheet Σ taking values in the pullback of T^*M by the map $\mathcal{X}$, $A_i = A_{i\mu}\mathrm{d}x^\mu$, $\mu = 1, 2$. More compactly, S may be regarded as a functional on the vector bundle morphisms from $T\Sigma$ to T^*M [3]. The action

JHEP07(2003)002

remains topological if, e.g., one adds to it the pullback of a 2-form B which has an exterior derivative $H = \mathrm{d}B$ that vanishes upon contraction with any hamiltonian vector field of $\mathcal{P}$ (cf. [4] for further details as well as for a topological generalization of the PSM associated to H-Poisson manifolds)[1]

$$\int_\Sigma \mathcal{X}^* B \,. \tag{1.2}$$

Likewise the local symmetries are not spoiled, if one adds e.g. a term of the form

$$\int_\Sigma C(X(x))\varepsilon \,, \tag{1.3}$$

where C is a Casimir function of $\mathcal{P}$ and ε a 2-form on Σ. (More generally, one may also add a sum of such terms, with several Casimir functions and volume forms). Due to the appearance of the 2-form(s) on Σ, the action is no longer topological in this case. Still, several features of the theory, including the structure of the moduli space of classical solutions considered in this paper, remain unaltered; consequently, the theories are called almost topological.

Poisson Sigma Models are of interest for at least three reasons. First, they provide a unifying framework for several two-dimensional field theories [1, 5, 6, 7, 8], including gravity and Yang-Mills gauge theories. Within this paper we will use part of this relation to check more general considerations.

The second point of interest in the topological Sigma Model (1.1) stems from its significance for the quantization of Poisson manifolds. It was noticed already early on [9, 5] (cf. also [10]) that the quantization of the two-dimensional field theory (1.1) is closely related to the quantization of the target manifold, interpreted as the collection of phase spaces for fictitious point particles (namely the symplectic leaves). Indeed, in a hamiltonian quantization, Σ taken cylindrical, any physical quantum state of the theory corresponds to a symplectic leaf (L, Ω) satisfying the integrality condition[2]

$$\oint_\sigma \Omega = 2\pi n \hbar \,, \quad n \in \mathbb{N} \,, \qquad \forall \sigma \in H^2(L) \,. \tag{1.4}$$

This relation is readily recognized as the condition for quantizability of the respective symplectic leaf L in the framework of Geometric Quantization (cf. e.g. [11]).

An approach to quantization applicable to general Poisson manifolds M is provided by the program of Deformation Quantization (cf. e.g. [12, 13]). Here the main idea is to find an *associative* (local) deformation of the product of functions on M in the form of a formal power series which in next to leading order in the deformation parameter coincides with the Poisson bracket. A solution to this by then already long-standing mathematical

[1] This corrects an inaccurate statement in [5], where only invariance of B under the hamiltonian vector fields generated by $\mathcal{P}$ was required. Likewise, if one adds a WZW-like H-term: invariance of the closed 3-form H would require the contraction (now a 2-form) to be closed only (instead of to be zero as turns out to be necessary for gauge invariance).

[2] In fact it is true in this form only if the respective symplectic leaf L is simply connected. Otherwise there are additional states, in general also corresponding to nonintegral leaves. Cf. [8] for a complete set of conditions.

JHEP07(2003)002

problem was provided by Kontsevich [14]. Kontsevich's formula, finally, received an illuminating interpretation [15] as appropriate two-point correlation function (evaluated on the boundary of a disc, $\Sigma \approx \mathbb{R}^2$) in a perturbation expansion of (1.1).

Last but not least, a Poisson Sigma Model may be regarded as an appropriate zero slope limit of (perturbative) String Theory in the background of a B-field. This may be a particularly interesting point of view in the context of open strings ending on D-branes, where in the case of a *constant* background, the effective Yang Mills theory induced on the D-brane was seen to become noncommutative [16, 17]. The induced noncommutative product was found to be the Moyal product [18] of the respective Poisson bivector $\mathcal{P}$ — the antisymmetric part of the inverse of the sum of B and the closed string metric g — which is the (previously known) specialization of the Kontsevich solution to the case of constant $\mathcal{P}$. One of the open issues in this realm is the generalization of this result to the case of nonconstant B-fields (resp. nonconstant bivectors $\mathcal{P}$). A reformulation of String Theory in terms of a nontopological deformation of (1.1) may provide the appropriate link in this context. Such a relation shall be pursued elsewhere, however.

In the present paper we focus on the moduli space $\mathcal{M}_{\rm cl}$ (denoted also more explicitly by $\mathcal{M}_{\rm cl}(\Sigma)$ or by $\mathcal{M}_{\rm cl}(\Sigma, M, \mathcal{P})$) of *classical* solutions of the (almost) topological models discussed above. For a fixed topology of the worldsheet (two-dimensional spacetime or base) manifold Σ and fixed target Poisson manifold $(M, \mathcal{P})$, we are interested in all smooth solutions to the classical field equations (stationary points of the action functional S), where solutions differing only by a gauge transformation (specified more clearly in section 2 below) are to be identified. For fairly reasonable topology of Σ (guaranteeing that rank $H_1(\Sigma)$ is finite) this moduli space will be finite dimensional. We will not be able to find globally valid solutions for completely arbitrary Poisson structures $(M, \mathcal{P})$, the main condition being the existence of a presymplectic form $\widetilde{\Omega}$ (on M or at least in a neighborhood of any symplectic leaf L of M) which is compatible with the Poisson bivector $\mathcal{P}$, i.e. whose pull back to a leaf L coincides with the symplectic structure of L as induced by $\mathcal{P}$ [19]. This result can, however, also be used to determine solutions if such a compatible presymplectic form only exists for some subset of leaves in $(M, \mathcal{P})$. Moreover, information about the space of solutions $\mathcal{M}_{\rm cl}(\Sigma, M, \mathcal{P})$ can still be found in more general cases by gluing techniques.

If the boundary of Σ is non-empty, there are some options as to what kind of boundary conditions to place on the fields and symmetries. In the present paper we allow for non-compact topologies of Σ, but will not restrict fields or symmetries at the (ideal) boundary. This may be motivated e.g. by physical models arising as particular PSMs. At least for some mathematical applications (but also e.g. in gravitational models in the hamiltonian formalism for "open spacetimes", cf., e.g., [8]) it is, however, also of interest to restrict some of the fields at the boundaries of Σ and simultaneously to freeze all or some of the symmetries there; the corresponding moduli space will in general differ from the above one and shall be denoted by $\mathcal{M}^0_{\rm cl}$ (irrespective of the precise kind of conditions at $\partial\Sigma$). For $\Sigma = [0,1] \times \mathbb{R}$ with boundary conditions requiring that the pull-back of A_i to $\{0\} \times \mathbb{R}$ and $\{1\} \times \mathbb{R}$ vanishes and frozen symmetries at these boundaries, $\mathcal{M}^0_{\rm cl}$ may be obtained also by symplectic reduction, and for sufficiently well-behaved $(M, \mathcal{P})$ it was found in [3] to carry the structure of a symplectic groupoid over M, which integrates the Lie algebroid on T^*M associated to the Poisson manifold M. In this case the transition from $\mathcal{M}_{\rm cl}^0$ to $\mathcal{M}$ essentially corresponds to an additional factorization, replacing M by the corresponding leaf space induced by $\mathcal{P}$. Many of our results can be adapted to the case of $\mathcal{M}_{\rm cl}^0(\Sigma)$ for any Σ, but we will not do so explicitly in the present paper.

The organization of the paper is as follows. In section 2 we discuss the field equations and symmetries of Poisson Sigma Models. We also briefly review how particular choices of Poisson structures result in known theories such as two-dimensional nonabelian gauge theories and the gravity theories mentioned above. In the subsequent section, section 3, we determine $\mathcal{M}_{\rm cl}(\Sigma, M, \mathcal{P})$ for particular choices of $(M, \mathcal{P})$ where methods other than the one presented in the main part of the paper are available such that there are results to be compared with. These are in particular the nonabelian gauge theories as well as Poisson Sigma Models with a topologically trivial foliation of the target manifold M, i.e. where M is fibered by leaves of trivial topology.

(Likewise results, not mentioned explicitely, are available for the case of the general Sigma Model when one restricts to topologically trivial worldsheet manifolds, $\Sigma \approx \mathbb{R}^2$. Furthermore, quite explicit results exist for the case of the 2d gravity models [7, 20, 21, 22]; due to an additional complication resulting from a nondegeneracy condition to be satisfied by the gravitationally acceptable solutions, these are mentioned only rather briefly here and a more detailed analysis is deferred to later work.)

In section 4 we present our main results concerning the solutions to the field equations of Poisson Sigma Models. They are summarized in theorem 2, providing the general solution for Poisson structures permitting a compatible presymplectic form in a neighborhood of any leaf, where the topology of Σ is arbitrary. We are also able to integrate the symmetries effectively, eqs. (2.5) below, such that we can provide representatives of any gauge equivalence class of the space of solutions. Leaves which do not have a neighborhood permitting a compatible presymplectic form are not covered by the theorem, but we provide a discussion of a possible generalization. (Let us mention right away that there are Poisson manifolds with prominent examples such as the Lie Poisson manifold $\mathfrak{g}^*$, $\mathfrak{g}$ compact semisimple, where there are *no* leaves permitting a compatible presymplectic form.)

In section 5 the results are specialized and compared to the results of section 3, in particular also to the two-dimensional nonabelian BF-theories with noncompact Lie algebra $\mathfrak{g}$. This example is also used in section 6 to illustrate the role of nonregular leaves. The analysis shows that in the case of BF-theory nonregular leaves are related to the irreducibility of connections. If the rank of the fundamental group of Σ is at most one, nonregular leaves contribute only a subset of lower dimension to the solution space $\mathcal{M}_{\rm cl}(\Sigma)$. For sufficiently large rank, the highest dimensional stratum of $\mathcal{M}_{\rm cl}(\Sigma)$ is obtained for irreducible connections corresponding to the origin as a nonregular leaf. The methods of this paper then give information on lower dimensional subsets of the solution space $\mathcal{M}$ with reducible connections. For Yang-Mills theories these are solutions with non-vanishing electric field, whereas an irreducible connection can only exist when the electric field vanishes. The results for Yang-Mills theories together with those of other methods indicate that the classification of solutions can be generalized straightforwardly to non-regular leaves, even though the explicit formula for the solution A in theorem 2 does not hold true.

JHEP07(2003)002

2. Setup

2.1 Field equations, gauge symmetries and moduli spaces

For a chosen local coordinate system in M, the fields can be understood as a collection of scalar fields X^i and 1-forms A_i, $i = 1, \ldots, n$, $n \equiv \dim M$, living on Σ. In general, this works on local patches of Σ only, however; still this perspective is sufficient for most of the purposes of the present paper. These patches on Σ may still be larger than those where local coordinates x^μ, $\mu = 1, 2$, on Σ exist and in which $A_i = A_{i\mu}(x)\mathrm{d}x^\mu$. (E.g. if M is chosen as the dual of a Lie algebra, X^i may be taken as linear coordinates on M. Then X^i and the 1-forms A_i exist globally on Σ, irrespective of the topology of the worldsheet manifold). For some purposes it is also convenient to consider A_i as a Grassmann odd field on Σ; in this context (such as in the field equations below) functional derivatives are understood as left derivatives always and, unless otherwise stated, products of forms (Grassmann objects) are understood to be wedge products.

The field equations of the action functional (1.1) are

$$\frac{\delta S}{\delta A_i} \equiv \mathrm{d}X^i + \mathcal{P}^{ij}(X)A_j = 0 \tag{2.1}$$

$$\frac{\delta S}{\delta X^i} \equiv \mathrm{d}A_i + \frac{1}{2}\mathcal{P}^{kl}{}_{,i}(X)\, A_k A_l = 0\,. \tag{2.2}$$

If the terms (1.2) and (1.3) are added only the second of these equations changes, since both terms depend on the field X only. Moreover, the contribution of (1.2) to the second equations, $\frac{1}{2}H_{ijk}\mathrm{d}X^j\mathrm{d}X^k$, vanishes upon use of (2.1) and the condition imposed on $H = \mathrm{d}B$! So, the addition of (1.2) or a similar WZW-term has neither an effect on the field equations nor on the local symmetries (cf. below) and consequently the classical moduli space is unchanged. A more interesting modification of the topological PSM is obtained when one drops the condition of a vanishing contraction of H with the bivector while simultaneously replacing the Jacobi identity for the Poisson bracket by the more general condition $\mathcal{P}^{ij}{}_{,s}\mathcal{P}^{sk} + \mathrm{cycl}(ijk) = \mathcal{P}^{ir}\mathcal{P}^{js}\mathcal{P}^{kt}H_{rst}$ characterizing an H-Poisson structure (cf. [4, 23, 24]); this changes both field equations and symmetries, but will not be further pursued in the present paper. The contribution from (1.3) to the lefthand side of (2.2), on the other hand, is simply $C_{,i}\,\varepsilon$ or, more generally, $C^\sigma{}_{,i}\,\varepsilon_\sigma$, where several Casimir functions C^σ and 2-forms ε_σ have been introduced, which explicitly breaks the topological nature of the equations:

$$\mathrm{d}A_i + \frac{1}{2}\mathcal{P}^{lm}{}_{,i}(X)A_l A_m + C^\sigma{}_{,i}(X)\varepsilon_\sigma = 0\,. \tag{2.3}$$

We remark in parenthesis that the field equations (2.2) are covariant with respect to target space diffeomorphism induced changes of field variables only if the field equations (2.1) are used. Given some auxiliary connection $\Gamma^i{}_{jk}$ on M, this may be cured by replacing (2.2) by

$$\mathrm{D}A_i + \frac{1}{2}\mathcal{P}^{kl}{}_{;i}(X)\, A_k A_l = 0\,, \tag{2.4}$$

where $\mathrm{D}A_i = \mathrm{d}A_i - \Gamma^j{}_{ik}\mathrm{d}X^k A_j$ (D is the induced exterior covariant derivative acting on forms taking values in the pullback bundle $\mathcal{X}^*T^*M$) and the semicolon denotes covariant differentiation with respect to Γ.

JHEP07(2003)002

Next we turn to the symmetries of the action (1.1). It is straightforward to check that under the infinitesimal symmetry transformations

$$\delta_\epsilon X^i = \epsilon_j \mathcal{P}^{ji}(X)\,, \qquad \delta_\epsilon A_i = \mathrm{d}\epsilon_i + \mathcal{P}^{kl}{}_{,i}\,(X) A_k \epsilon_l\,, \tag{2.5}$$

where the ϵ_i are arbitrary functions on Σ, the action changes only by a total divergence $\int_\Sigma \mathrm{d}(\epsilon_i \mathrm{d}X^i)$ thanks to the Jacobi identity

$$\mathcal{P}^{il}\mathcal{P}^{jk}{}_{,l} + \mathcal{P}^{jl}\mathcal{P}^{ki}{}_{,l} + \mathcal{P}^{kl}\mathcal{P}^{ij}{}_{,l} = 0 \tag{2.6}$$

for the Poisson tensor. Almost topological models have the same symmetries due to the definition of a Casimir.

The set of gauge transformations (2.5) is (in general slightly over-)complete, or, in other words, it is an (in general slightly reducible) generating set of gauge transformations (cf. [25] for definitions and further details).[3] This in particular implies that *any* other local (gauge) symmetry of (1.1) (invariance of the functional, parametrized by some set of arbitrary functions on Σ) can be expressed in terms of (2.5) up to so called trivial gauge transformations *and* with possibly field dependent parameters ϵ_i. If y^α denotes the set of all fields of the action functional S, $S = S[y^\alpha]$, α being a collective index, then trivial gauge transformations are (infinitesimally) of the form $\delta_\mu y^\alpha = \mu^{\alpha\beta}(\delta S/\delta y^\beta)$ (the sum also involving an integration) for some graded "antisymmetric" but otherwise arbitrary μ. They are called trivial because on-shell (that is on the space of solutions to the field equations) they act trivially and because they exist for any action functional S. According to theorem 17.3 of [25] *any* symmetry of (1.1) vanishing on-shell is of this form.

The trivial transformations form a normal subgroup $\mathcal{N}$ of all the gauge transformations $\bar{\mathcal{G}}$. It is only the respective quotient group $\mathcal{G} = \bar{\mathcal{G}}/\mathcal{N}$ that is of relevance for the gauge identification of solutions. The infinitesimal gauge transformations certainly form a Lie algebra (the infinite-dimensional Lie algebra of $\bar{\mathcal{G}}$). However, the representatives (2.5) do not; instead one finds:

$$[\delta_\epsilon, \delta_{\tilde\epsilon}] = \delta_{[\epsilon,\tilde\epsilon]} + \int_\Sigma \epsilon_j \tilde\epsilon_i \mathcal{P}^{ij}{}_{,kl}\, \frac{\delta S}{\delta A_k}\frac{\delta}{\delta A_l}\,, \tag{2.7}$$

where $[\epsilon, \tilde\epsilon]_k \equiv \epsilon_i \tilde\epsilon_j \mathcal{P}^{ij}{}_{,k}\,(X)$. This is of the expected form since the commutator of two gauge transformations is another one, and the representatives (2.5) are complete. Also the (field dependent) coefficient in front of the contribution vanishing on-shell is indeed symmetric in the Grassmann part. (But even in the absence of the latter contribution, making the algebra of (2.5) an "open" one, the field dependence of the new parameter $[\epsilon,\tilde\epsilon]_i$ spoils the Lie algebra property.) Completeness ensures also that the obvious worldsheet diffeomorphism invariance of (1.1) can be expressed in terms of (2.5); indeed for any generating

[3] At least for topologies of Σ with $\operatorname{rank} H_1(\Sigma) \leq 1$ this is obvious from a hamiltonian analysis of the theory, cf. e.g. [1]. For general topologies of the worldsheet it may, strictly speaking, require a separate proof. In any case, we will consider as symmetries to be factored out all those that are generated by (2.5).

vector field $\xi \in \Gamma(T\Sigma)$ one finds for the respective Lie derivative acting on the space of fields under consideration

$$\mathcal{L}_\xi = \delta_{\langle A,\xi\rangle} + \int_\Sigma \langle \frac{\delta S}{\delta A_i}, \xi\rangle \frac{\delta}{\delta X^i} - \int_\Sigma \langle \frac{\delta S}{\delta X^i}, \xi\rangle \frac{\delta}{\delta A_i} \approx \delta_{\langle A,\xi\rangle}\,, \tag{2.8}$$

where the latter weak equality sign $\approx$ is used to denote on-shell equality.

The moduli space $\mathcal{M}_{\rm cl}$ is now defined as the space of all gauge inequivalent smooth fields $X^i(x)$ and $A_{i\mu}(x)\mathrm{d}x^\mu$ on a fixed worldsheet Σ, which we denote collectively by Φ, satisfying the field equations (2.1) and (2.2):

$$\mathcal{M}_{\rm cl}(\Sigma) = \frac{\{\Phi \mid \delta S/\delta\Phi = 0\}}{\text{gauge equivalence}}\,, \tag{2.9}$$

where "gauge equivalence" is the equivalence relation generated by (2.5); in particular the gauge group, called $\mathcal{G}$ above, is taken to be connected and simply connected. According to (2.8), on the space of solutions infinitesimal diffeomorphsims are generated by the symmetries under consideration; correspondingly, *at least* the component of unity of Diff(Σ) will be factored out in (2.9). It may also happen, however, that several disconnected components of Diff(Σ) fit into $\mathcal{G}$, and thus are factored out in $\mathcal{M}_{\rm cl}$ — [26] provides an explicit example for this scenario.

Note that in the general case the gauge symmetries are known explicitly only in their infinitesimal form (2.5). To determine whether two given solutions are gauge equivalent, we will, however, have to integrate these symmetries in one way or another.

According to Noether's second theorem the existence of nontrivial gauge symmetries implies dependencies among the field equations. With (2.5) one finds

$$\mathcal{P}^{ij}(X)\,\frac{\delta S}{\delta X^j} \equiv \mathcal{P}^{ij}{}_{,k}(X) A_j \frac{\delta S}{\delta A_k} + \mathrm{d}\frac{\delta S}{\delta A_i}\,. \tag{2.10}$$

A discussion of the relevance of Noether's identities in the BV formalism with a special emphasis on the example of the PSM may be found in [27].

As mentioned already in the Introduction, in the case that Σ has (ideal) boundary components, one possibly wants to impose some additinal boundary conditions on the admissible fields Φ and on the parameters ϵ_i in (2.5). Acceptable boundary conditions on Φ should be such that $\int_{\partial\Sigma} A_i \delta X^i$ vanishes, so that (1.1) has well-defined functional derivatives. This is guaranteed e.g. by the Cattaneo-Felder boundary conditions $\langle A_i, v\rangle(x) = 0$ for any $x \in \partial\Sigma$ and $v \in T_x\partial\Sigma \subset T_x\Sigma$. Alternatively, one may also consider some Dirichlet-type boundary conditions on $\mathcal{X}$ (more generally, some mixture of both types of conditions, cf. e.g. [8]), where the image of $\partial\Sigma$ has to lie within one symplectic leaf of $(M, \mathcal{P})$ so as to permit nontrivial solutions of the field equations.

Note that the moduli space (2.9) does not change if the symmetries are restricted *correspondingly* on the boundary, i.e. if any gauge orbit of the unrestricted theory leading to the moduli space $\mathcal{M}_{\rm cl}$ is in 1-1 correspondence with a gauge orbit of the restricted gauge equivalence on the set of the restricted fields. E.g. in the particular model (1.1) with $\mathcal{P} \equiv 0$ and upon the choice of the CF boundary conditions this would imply that ϵ_i should be constant along the boundary. Requiring, instead, ϵ_i to vanish on all of $\partial\Sigma$ then enlarges the moduli space to a bigger one, $\mathcal{M}^0_{\rm cl}$, which yields $\mathcal{M}_{\rm cl}$ only after taking another quotient.

JHEP07(2003)002

In the gravitational context there will be also other moduli spaces of relevance, denoted collectively by $\mathcal{M}_{\rm cl}^{\rm grav}(\Sigma)$; we will discuss this briefly below.

As alluded to already above, not all of the symmetry generators (2.5) are independent, at least if the topology of Σ is nontrivial and $\mathcal{P}$ has a nontrivial kernel. For illustration let us consider a Poisson tensor for which the first k coordinates X^i, $i = 1, \ldots, k$, are Casimir functions. Then, for any choice of ϵ_i nonvanishing only in the first k indices, one has $\delta_\epsilon X^i = 0$ and $\int_\alpha \delta_\epsilon A_i = \int_\alpha \mathrm{d}\epsilon_i = 0$ for any closed loop α. This corresponds to k times rank $H^1(\Sigma)$ nontrivial (global) relations between the generators (2.5). In a BRST or BV quantization scheme this requires ghosts for ghosts (for an explicit construction of the respective hamiltonian BRST charge for $\Sigma = S^1 \times \mathbb{R}$ cf. [1]). This complication is e.g. absent if Σ is a disc as in [15] since H^1 vanishes in this case.

2.2 Special cases

As mentioned already in the Introduction, there are several particular cases of models of the type (1.1) for which $\mathcal{M}_{\rm cl}(\Sigma)$ (or at least some facts about $\mathcal{M}_{\rm cl}(\Sigma)$ like its dimension) is known or can be determined by other means.

2.2.1 Two-dimensional nonabelian gauge theories

The most obvious of these is the specification to a nonabelian gauge theory resulting from a choice of $(\mathcal{P}^{ij})$ linear in X, $\mathcal{P}^{ij} = f^{ij}{}_k X^k$. In this way, $M \equiv \mathfrak{g}^*$ can be identified with the dual of the Lie algebra defined by the structure constants $f^{ij}{}_k$, equipped with the Kirillov-Kostant Poisson structure defined by $\mathcal{P}_X(\alpha, \beta) = X([\alpha, \beta])$ where $X \in \mathfrak{g}^*$ and $\alpha, \beta \in T^*_X M \equiv \mathfrak{g}^{**} \equiv \mathfrak{g}$. In coordinates $X = X^i T_i$ with generators T_i of $\mathfrak{g}^*$, this yields the above linear Poisson tensor. In what follows, we will restrict ourselves to semisimple Lie algebras such that the dual of $\mathfrak{g}$ can be identified with $\mathfrak{g}$ itself by means of the Cartan-Killing metric, which we will denote by tr below. We then express the fields as the Lie algebra valued functions $A = A_i T^i$ and $X = X^i T_i$ on Σ for which the field equations (2.1), (2.2) may be rewritten as $F \equiv \mathrm{d}A + A \wedge A = 0$ and $D_A X \equiv \mathrm{d}X + [A, X] = 0$. Furthermore, the gauge symmetries (2.5) can be integrated to the equivalence relations $A \sim A^g \equiv g^{-1}Ag + g^{-1}\mathrm{d}g$, $X \sim X^g \equiv g^{-1}Xg$ where $g(x)$ is an arbitrary G-valued function on Σ. We restrict ourselves to trivial bundles $\Sigma \times G$; for nontrivial bundles, already $\mathcal{X}$ is no more just a map $\Sigma \to M$, and much of what has been said above needs reformulation (cf. [1, 28] for such an attempt).

The gauge transformations deserve further mention: $g(x)$ denotes a map from the worldsheet Σ to the chosen structure group G whose Lie algebra $\mathfrak{g}$ has structure constants $f^{ij}{}_k$. The choice of G is not unique, different choices differing by the fundamental group $\pi_1(G)$. A nontrivial $\pi_1(G)$, on the other hand, gives rise to "large" gauge transformations for a multiply connected worldsheet Σ, i.e. gauge transformations not being connected to the identity and thus not resulting from a direct integration of the infinitesimal form of the symmetries. Thus, for a better comparison, G should always be taken to be the uniquely determined simply connected group having the given structure constants $f^{ij}{}_k$.[4]

[4] In some cases, as e.g. for the Lie algebra $sl(2, \mathbb{R})$, this excludes the possibility of a matrix representation of G. It is, however, still possible to construct the moduli space along the lines below.

JHEP07(2003)002

Yang-Mills theories are obtained by adding the part (1.3) with the quadratic Casimir $C(X) = \mathrm{tr}(X^2)$ and a volume form ε on Σ to the action. The field equation for X is unchanged while the zero curvature condition turns into $F = -C_{,i} T^i \varepsilon = -X\varepsilon$, rendering X to play the role of the electric field. As in the general case, gauge transformations are not changed by adding (1.3).

2.2.2 Two-dimensional gravity models

Let us choose $M \simeq \mathbb{R}^3$ with coordinates $(X^i) := (X^+, X^-, \phi) \in \mathbb{R}^3$ and a Poisson bracket defined through $\{X^+, X^-\} = W(\phi)$, $\{X^\pm, \phi\} = \pm X^\pm$ for a (sufficiently) smooth function W. Such a bracket has one Casimir function C given by[5]

$$C = 2X^+X^- - 2\int^\phi W(z)\mathrm{d}z\,. \tag{2.11}$$

Interpreting X^a, $a \in \{+,-\}$, as a Lorentz vector in a two-dimensional Minkowski space, the bracket $\{\cdot,\cdot\}$ is seen to be invariant with respect to the corresponding Lorentz transformations. Actually, for this purpose and most of what is described below, W could be allowed also to depend on the Lorentz invariant combination X^+X^-; for simplicity, however, we restrict ourselves to functions of the third coordinate ϕ only.

Identifying the respective 1-forms $A_\pm$ with a zweibein $e_\pm = e^\mp$ (using the Minkowski metric of a frame bundle to raise and lower indices) and A_ϕ with a spin connection 1-form ω ($\omega^a{}_b = \varepsilon^a{}_b\,\omega$, ε antisymmetric and normalized according to $\varepsilon^{+-} = 1$), and upon dropping a surface term, the action (1.1) assumes the form

$$S^{\mathrm{grav}} = \int_\Sigma X_a De^a + \phi\mathrm{d}\omega + W(\phi)\,\varepsilon\,, \tag{2.12}$$

where $De^a \equiv \mathrm{d}e^a + \omega^a{}_b \wedge e^b$ is the torsion 2-form and $\varepsilon = e_+ \wedge e_-$ is the (*dynamical*) volume form on Σ. Thus S is seen to yield an action for a gravitational theory defined on a two-dimensional spacetime Σ.

In fact, e.g. if W is a convex function, we may eliminate the fields X^i by means of their algebraic field equations, and the action may be seen to take the purely geometrical form

$$S^{\mathrm{geom}}[g] = \int_\Sigma \mathrm{d}^2x\sqrt{-g}f(R)\,, \tag{2.13}$$

where f is the Legendre transform of $-2W$ and R is the Ricci scalar of the torsion free Levi Civita connection associated to the two-dimensional metric $g_{\mu\nu} = 2e_\mu{}^+e_\nu{}^-$. The prototype of such a higher derivative theory is provided by R^2 gravity, $f(R) = \frac{1}{8}R^2 + 2$, resulting from the quadratic Poisson bracket defined by $W = 1 - \phi^2$.

There are also more general possibilities to obtain gravitational models as particular PSMs. We will come back to this elsewhere [29].

[5]To fully define C by this equation, one may choose some fixed constant for the lower bound of the integral.

3. Results by other methods

In special cases of the Poisson manifold $(M, \mathcal{P})$, there are known methods to solve the field equations which will be recalled here for later use.

3.1 $\mathcal{M}_{\rm cl}(\Sigma)$ for topologically trivial Poisson manifolds

Let us assume within this subsection that the Poisson manifold $(M, \mathcal{P})$ is foliated regularly into symplectic leaves and that the typical leaf admits a set of *globally* defined Darboux coordinates X^α. Indexing the leaves by (possibly only locally defined) coordinates X^I, $I = 1, \ldots, k$, (any set of independent Casimir functions on M where k is the dimension of the kernel of $\mathcal{P}$, which, by assumption, is constant all over M), there thus exists a combined coordinate system $(X^i) = (X^I, X^\alpha)$ on M for which the matrix $\mathcal{P}^{ij}$ is zero everywhere except for the block $\mathcal{P}^{\alpha\beta}$ which has standard Darboux form.

In such a coordinate system the field equations (2.1), (2.2) are trivial to solve: Clearly, the first set of equations implies that the $X^I(x)$ are constant functions all over Σ and that the A_α are determined uniquely by means of $X^\alpha(x)$; if $\Omega_{\alpha\beta}$ denotes the (constant) inverse to $\mathcal{P}^{\alpha\beta}$, $A_\alpha = \Omega_{\beta\alpha} \mathrm{d}X^\beta$. The nontrivial content of the second set of the equations, furthermore, reduces to $\mathrm{d}A_I = 0$. Also the local symmetries (2.5) can be integrated easily in this case. One learns that any smooth choice of the set of functions $X^\alpha(x)$ is gauge equivalent to any other choice and that any A_I just transforms like a U(1) gauge field, i.e. $A_I \sim A_I + \mathrm{d}h_I$ for any (smooth) function $h_I(x)$.

Thus, restricting M further to $\mathbb{R}^n$ for simplicity, we obtain

Proposition 1. *Let $(\mathbb{R}^n, \mathcal{P})$ be a trivially foliated Poisson manifold with leaves homeomorphic to $\mathbb{R}^{n-k}$ where $k = \dim\ker\mathcal{P}$. A set of representatives of the gauge equivalence classes of solutions to the field equations is given by*

$$X^I(x) = C^I = \text{const.}\,, \qquad X^\alpha(x) = 0 = A_\alpha\,, \qquad A_I = \alpha_I \quad \text{with} \quad [\alpha_I] \in H^1(\Sigma)\,. \tag{3.1}$$

Correspondingly, the classical moduli space is found to be of the form:

$$\mathcal{M}_{\rm cl}(\Sigma) = \mathbb{R}^{k(r+1)} \qquad \text{with} \qquad r \equiv \operatorname{rank} H^1(\Sigma) = \operatorname{rank} \pi_1(\Sigma) \quad \text{and} \quad k = \dim\ker\mathcal{P}\,. \tag{3.2}$$

Here, k is also the (by assumption constant) codimension of the symplectic leaves in M. As usual, $H^1(\Sigma)$ denotes the (always abelian) first cohomology of the base manifold Σ, its rank being the number of independent generators.

Note that in the original formulation of the action (1.1) admitting the above mentioned coordinates on M, as obtained, e.g., in the case of a gravity model, the matrix $\mathcal{P}$ as a function on M may be highly nonlinear. Then the change of coordinates implicitly used above provides a trivialization of the otherwise possibly not that simple field equations (for a partial illustration of this point cf. [7]).

In the above we assumed that the symplectic foliation of M is topologically trivial while Σ was kept arbitrary. Alternatively, one may achieve similar results (cf., e.g., [1, 7]) for the case that one restricts attention to a local, topologically trivial patch of Σ. The reason for this is that any (generic) point in M has a neighborhood which admits Casimir-Darboux coordinates putting $\mathcal{P}$ in the above mentioned standard form [30]. It is the intention of the present paper to extend these results to the case where Σ is left completely arbitrary, but now M need not necessarily be foliated by symplectic leaves homeomorphic to a linear space (as implied by the global existence of Darboux coordinates). E.g., we will be able to cover cases such as those where M is (regularly) foliated by topologically nontrivial leaves provided only that they have trivial second cohomology. For specialization to topologically trivial leaves the above results will then be regained.

3.2 Gauge theories

It is precisely in the case of linear Poisson structures in the setting of section 2.2.1 that the second set of field equations (2.2) decouples from the first set (2.1), being a set of equations for A only. The moduli space of solutions to $F = 0$ (the space $\mathcal{A}_0$ of flat connections) modulo gauge symmetries is well known: a flat connection is characterized uniquely by its holonomies (elements of G) along a set of representatives for generators of $\pi_1(\Sigma)$, and gauge transformations act thereon by joint conjugation. Therefore, the space of flat connections modulo gauge transformations is given by $\mathrm{Hom}(\pi_1(\Sigma), G)/\operatorname{Ad}_G$ where Ad_G denotes the adjoint action of G, and any gauge invariant function on the space of flat connections can be written as a function of traces of holonomies, called Wilson loops, along generators of the fundamental group. Note that for a compact structure group G the space of flat connections modulo gauge transformations is compact. This space is in general not a manifold but only a stratified space because the adjoint action may have fixed points on $\mathcal{A}_0$. The smooth part of $\mathcal{M}_{\mathrm{cl}}(\Sigma)$ is given by the space of gauge equivalence classes of those flat connections which yield holonomies with centralizer in G^r, $r \equiv \operatorname{rank} \pi_1(\Sigma)$, of minimum dimension. If r is large enough, these are irreducible connections; the latter can be defined as connections for which there is no non-central element $g \in G \backslash Z(G)$ commuting with all holonomies.

Given a flat connection A, one can determine X by solving the equation $\mathrm{d}X = -[A, X]$, which locally can be integrated to $X(x) = \operatorname{Ad}_{g_{x_0,x}} X(x_0)$ where x_0 is an arbitrary fixed point in Σ and $g_{x_0,x}$ denotes the parallel transport between x_0 and x along an arbitrary path. In a simply connected neighborhood of x_0, the parallel transport $g_{x_0,x}$ is unique (i.e., independent of the path) because A is flat, and so X is well defined. If Σ is not simply connected, the initial value $X(x_0)$ has to fulfill $r \equiv \operatorname{rank} \pi_1(\Sigma)$ integrability conditions because $X(x_0) = \operatorname{Ad}_{h_{x_0}} X(x_0)$ for any holonomy h_{x_0} along a closed curve based in x_0. Given an irreducible flat connection A, evidently these conditions cannot be fulfilled non-trivially for a semisimple Lie group G, $X \equiv 0$ being the only admissible solution in this case.

Only for a reducible flat connection A can there be non-trivial solutions X. The explicit form, in some gauge, is determined by a given connection A via the field equation $\mathrm{d}X = -[A, X]$, which infinitesimally expresses the fact that a change in X is given by conjugation. Therefore, the image of Σ under the map $\mathcal{X} \colon \Sigma \to \mathfrak{g}$ corresponding to X has to lie entirely within an adjoint orbit of $\mathfrak{g}$, the explicit form being determined by a particular flat connection. If only "small" gauge transformations are allowed, gauge

JHEP07(2003)002

transformations acting on X correspond to smooth deformations of the map $\mathcal{X}$ within an adjoint orbit, demonstrating that gauge equivalence classes of solutions X are given by particular homotopy classes of maps from Σ to some adjoint orbit in $\mathfrak{g}$.

Given a semisimple Lie Group G of rank k, there are always irreducible flat connections on the Riemann surface Σ *provided* that $\operatorname{rank}\pi_1(\Sigma)$ is large enough. If Σ has genus g and n holes (boundary components), the fundamental group can be represented by $2g+n$ generators $a_1,\ldots,a_g$, $b_1,\ldots,b_g$ and $m_1,\ldots,m_n$ with one relation $[a_1,b_1]\cdots[a_g,b_g]m_1\cdots m_n=1$ using the commutator $[a,b]:=aba^{-1}b^{-1}$. A flat connection is uniquely specified by its $2g+n$ holonomies in G around the generators, which also have to be subject to the given relation (eliminating one free holonomy). Furthermore, factoring out gauge transformations generically eliminates $\dim G$ parameters for choosing the holonomies. Thus, we have a maximal dimension $(\operatorname{rank}\pi_1(\Sigma)-2)\dim G=(2g+n-2)\dim G$ for the space of irreducible flat connections modulo gauge transformations (see also [31]). This shows that $\operatorname{rank}\pi_1$ has to be larger than two in order to allow irreducible connections.

Thus, in general the moduli space $\mathcal{M}_{\rm cl}(\Sigma)$ of gauge equivalence classes of flat connections has a smooth stratum of maximal dimension consisting of gauge equivalence classes of irreducible flat connections (if G is non-compact, this stratum may be non-Hausdorff, but one can choose a dense subspace which is Hausdorff) of dimension $(\operatorname{rank}\pi_1(\Sigma)-2)\dim G$, whereas reducible flat connections give lower-dimensional strata [31].

Solutions for X which lie in adjoint orbits of maximal dimension (which is $\dim G-k$; this case corresponds to a flat connection with holonomies generating a maximal abelian subgroup of G) contribute a subset maximally of dimension $k(\operatorname{rank}\pi_1(\Sigma)+1)$. The first part, i.e. $k\operatorname{rank}\pi_1(\Sigma)$, is the dimensionality of the space of connections whose holonomies generate a maximal abelian subgroup of G, whereas the contribution of dimension k is the remaining freedom in choosing X (which is only free at a single point and has to commute with all holonomies, i.e. it must also lie in the maximal abelian subgroup of dimension k). Clearly this is of lower dimension if the rank of the fundamental group is large enough. The case of small $\operatorname{rank}\pi_1(\Sigma)$ is special; in particular if $\operatorname{rank}\pi_1(\Sigma)\leq 1$, *all* flat connections are reducible and a dense set in $\mathcal{M}_{\rm cl}(\Sigma)$ is provided by connections which lead to non-trivial X-solutions; this is also the case if $\pi_1(\Sigma)$ is abelian (e.g., if Σ is a torus).

For Yang-Mills theories, the curvature of A is not necessarily zero but given by $F=-X\epsilon$ in terms of the electric field X. Therefore, the above strategy cannot be applied for a general solution since holonomies are no longer invariant under deformations of the curve defining $g_{x_0,x}$. This is the case only if X vanishes where we have the same solutions as described above given by flat connections. But in the physically more interesting case of non-zero electric field X, a connection cannot be flat and solutions for A have to be determined by other means. This shows that differences between BF- and Yang-Mills theories only arise in the sector of non-vanishing X which in BF-theories leads to reducible flat connections and in Yang-Mills theories to non-flat connections. The standard methods reviewed in the present subsection start by using the mathematically well-studied space of irreducible flat connections and are insensitive to those differences. On the other hand, we will see that the methods of this paper are well-suited to determine solutions with non-vanishing X-field and nicely demonstrate the key difference between solutions to both theories.

3.3 Gravity models

In the gravitational setting, one is interested in maximally extended solutions to the field equations resulting from a variation of the action (2.12) for fixed topology of Σ, having a globally smooth and nondegenerate metric, and identifying solutions which are mapped into one another by the gravitational symmetries, i.e. by diffeomorphisms and local Lorentz transformations. This program has been carried out in full generality in [20, 21, 22], yielding implicitly a description of the "gravitational moduli space" $\mathcal{M}_{\mathrm{cl}}^{\mathrm{grav}}(\Sigma)$, which, in particular, was found to be finite dimensional for any fixed topology of Σ.

With the methods of the present paper we will be able to derive information about the space of solutions at a global level. Because the requirement of a non-degenerate metric (including subtle relations of the gravitational symmetries to the symmetries generated by (2.5) — cf. also (2.8) and the explicit discussion of this relation in [26]) needs some care, however, we will discuss this application elsewhere. Also other issues which are specific to gravity, such as the completeness of the resulting two-dimensional space-time, can be studied within the present setting.

4. Moduli space of classical solutions

In this section we present our results concerning solutions to the field equations of Poisson Sigma Models. The topology of the worldsheet Σ is taken to be fixed. Along possible boundary components of Σ neither the fields nor the local symmetries will be restricted in this context; for given Σ we look for the moduli space of solutions to the field equations (2.1), (2.2) subject to the equivalence relation generated by the symmetries (2.5). The situation remains unchanged when boundary conditions on the fields are added in the same "number" as symmetries along the boundaries are frozen. In a hamiltonian formulation of the model on $\Sigma \cong \mathbb{R} \times \mathbb{R}$, this is not always adequate or in the line of a particular (physical or mathematical) problem. Examples for this are two-dimensional gravity models on open spacetimes as well as the recent considerations of Cattaneo and Felder: In both cases additional parameters of the moduli space appear by freezing *all* the symmetries on the boundary of the first factor $\mathbb{R}$ of Σ while only part of the fields are subject to boundary conditions (in the case of Cattaneo and Felder e.g. only the tangential components of A_i). The present method, however, may be applied also to such cases.

4.1 Solutions for $\mathcal{X}$

For the case that $\mathcal{P}$ is linear, it was found in section 2.2 that it is advisable first to solve the equations (2.2) for the A-fields, as (for the topological model) the field equations for them decouple from the fields $X(x)$. In the general case, however, the field equations (2.2) are much harder to solve and in the present paper this will be achieved only under certain conditions on the target manifold $(M, \mathcal{P})$. The field equations (2.1) for the X-fields, on the other hand, can be solved in full generality for arbitrary target, and even the local symmetries may be integrated easily (although in a rather abstract manner):

Theorem 1. *Let $(\mathcal{X}\colon \Sigma \to M, A)$ be a solution to the field equations (2.1) and (2.2).*

Then the image of $\mathcal{X}$ lies entirely within one of the symplectic leaves $L \subset M$ of the foliation of M. All gauge equivalence classes of solutions $\mathcal{X}$ are provided by the homotopy classes of maps from Σ to any L.

Proof: Let $X \in \mathcal{X}(\Sigma)$ be a point in the image of $\mathcal{X}$. We first assume that X is a regular point of the foliation of M into symplectic leaves, i.e. that X lying in a symplectic leaf L has a neighborhood $\mathcal{U}$ homeomorphic to $(\mathcal{U} \cap L)_X \times \mathbb{R}^k$ with $k = \dim \ker \mathcal{P}(X)$ where $(\mathcal{U} \cap L)_X$ denotes the connected component of $\mathcal{U} \cap L$ containing X (only in the case of a leaf L which lies densely in a part of M do we have $(\mathcal{U} \cap L)_X \neq \mathcal{U} \cap L$ for all $\mathcal{U}$). After choosing local coordinates in $\mathcal{U}$ adapted to the decomposition into $\mathcal{U} \cap L$ and $\ker \mathcal{P} \cong \mathbb{R}^k$, it is immediate to see that the components of $\mathcal{X}$ along $\ker \mathcal{P}$ have to be constant in $\mathcal{U}$ owing to eq. (2.1). Therefore, the image of $\mathcal{X}$ lies in L in a neighborhood of any regular point and the first assertion follows for the case of a regular foliation of M.

In general, however, M is not foliated regularly into symplectic leaves, meaning that there are also lower dimensional leaves which then lie in the boundary of a higher dimensional one. For X lying in a lower dimensional leaf $L \subset \partial L'$, where L' is a higher dimensional one, the above reasoning shows that all derivatives of components of adapted coordinates "normal" to L have to vanish. But this information alone is not sufficient to ensure that the image of $\mathcal{X}$ lies entirely in L, for there are directions normal to L but tangential to L' leading to derivatives of $\mathcal{X}$ which have to vanish only in L, not in L'. It is then possible to construct smooth maps $\mathcal{X}\colon \Sigma \to M$ which connect L' with L. Using the complete field equations (2.1) for $\mathcal{X}$, however, we can exclude such maps thanks to the uniqueness theorem for solutions of first order differential equations: Suppose there is a solution $\mathcal{X}$ of (2.1) connecting L' with L and a corresponding smooth solution A of (2.2). We can then find a smooth path c of finite parameter length in Σ such that the interior of $\mathcal{X}(c)$ lies in L' and its endpoint in L. Eq. (2.1) then implies that the restriction of $\mathcal{X}$ to c is the integral curve of a smooth vector field (determined by A) vanishing at the endpoint of $\mathcal{X}(c)$. The existence of such an integral curve reaching the singularity of the smooth vector field in a finite parameter distance is a contradiction. This proves our first assertion in the general case.

Using again local coordinates adapted to the foliation, it is easy to see that the gauge transformations (2.5) are infinitesimal homotopies of the map $\mathcal{X}\colon \Sigma \to M$, immediately leading to the second assertion. $\square$

Since in the proof we only used the field equation (2.1) and symmetries (2.5) for X and the fact that A is subject to a first order differential equation which also holds true for an almost topological model, we obtain the following

Corollary 1. *Let $(\mathcal{X}\colon \Sigma \to M, A)$ be a solution to the field equations (2.1) and (2.3).*

Then the image of $\mathcal{X}$ lies entirely within one of the symplectic leaves $L \subset M$ of the foliation of M. All gauge equivalence classes of solutions $\mathcal{X}$ are provided by the homotopy classes of maps from Σ to any L.

JHEP07(2003)002

Thus, although the model is no longer topological with the term (1.3), the X-solutions are still classified solely by topological properties of the spaces Σ and M (the latter as a foliated space).

4.2 Compatible presymplectic forms

The main tool for constructing solutions for A corresponding to a map $\mathcal{X}$ into a leaf L will be a presymplectic form $\widetilde{\Omega}$, which, upon restriction to tangential vectors to any leaf L' in U coincides with the respective symplectic 2-form $\Omega_{L'}$ induced by the given Poisson bracket on M. Such a *compatible presymplectic form* $\widetilde{\Omega}$ of $\mathcal{P}$ does not exist under all circumstances and if it exists, it will not be unique.

General conditions for the existence of such a 2-form $\widetilde{\Omega}$, compatible with $\mathcal{P}$ in the above sense, have been investigated in [32] where the obstruction has been identified as the characteristic form class of the Poisson bivector $\mathcal{P}$ and recently in [19] under the condition that there is an integrable distribution transversal to the leaves in U, where they have been put into the form of descent equations. Specializing these equations to particular cases gives the following results which we will use later:

Corollary 2. *If M is foliated trivially, i.e. it is of the form $M \cong L \times \mathbb{R}^k$, then a necessary condition for the existence of a compatible presymplectic form in a neighborhood of L is*

$$\partial_I \oint_\sigma \Omega_L = 0$$

where ∂_I denotes any differentiation transversal to L and σ is a closed 2-cycle in L. This means that the symplectic volume of any closed 2-cycle in a leaf has to be constant in M.

The second result, which can be easily verified, applies to leaves of trivial second cohomology:

Lemma 1. *If Ω_L has a symplectic potential θ_L on any leaf L in $M \cong L \times \mathbb{R}^k$, i.e. $\Omega_L = \mathrm{d}_{||}\,\theta_L$, and θ_L varies smoothly from leaf to leaf, then $\widetilde{\Omega} := \mathrm{d}\theta$ is a compatible presymplectic form on M.*

In particular, if all leaves L in a trivially foliated $M \cong L \times \mathbb{R}^k$ have trivial second cohomology, then there exists a compatible presymplectic form on M.

The notation here is as follows: Ω_L and θ_L are differential forms on the leaves which depend parametrically on coordinates transversal to the leaves (e.g. Casimir functions). The derivative operator $\mathrm{d}_{||}$ only acts on coordinates inside the leaf L, whereas d is the exterior derivative in the embedding space M and acts on all coordinates; θ, finally, by definition coincides with θ_L on any leaf L.

As already mentioned, a compatible presymplectic form is not unique. Given one such form $\widetilde{\Omega}$, one can always add a closed 2-form λ which vanishes when pulled back to the leaf, giving in fact the complete freedom in defining $\widetilde{\Omega}$ [32, 19].

It is interesting to observe that conditions such as in corollary 2 have been found as obstructions to integrate the Lie algebroid T^*M to a smooth Lie groupoid, cf. [33, 34].

JHEP07(2003)002

4.3 Solutions for A

We now may solve for the A-fields assuming a fixed map $\mathcal{X}$ which in particular singles out a symplectic leaf L. (To obtain all solutions, all possible leaves L as well as representatives of all homotopy classes from maps $\mathcal{X}$ to L have to be considered; in any of these cases we then proceed by solving for A for given $\mathcal{X}$.) In general, the field equation for A is harder to solve, and we will start our discussion with a special case and later discuss generalizations.

4.3.1 Solutions for topological models corresponding to regular leaves of trivial holonomy

We first assume that we are dealing with a map $\mathcal{X}$ into a regular leaf of trivial holonomy which allows a compatible presymplectic form in a neighborhood. By this assumption, we can choose a set of Casimir functions C^I in the neighborhood such that L is given by the preimage of zero (and L has a neighborhood of the form $L \times \mathbb{R}^k$). We then arrive at

Proposition 2. *For a given map $\mathcal{X}$ with image in a symplectic leaf L of trivial holonomy which has a neighborhood U permitting a presymplectic form $\widetilde{\Omega}$ compatible with $\mathcal{P}$, any solution to the field equations (2.1), (2.2) may be written in the form*

$$A_i = -\mathcal{X}^*(\partial_i \lrcorner \widetilde{\Omega}) + \alpha_I \, \mathcal{X}^*(\partial_i C^I)\,, \tag{4.1}$$

where C^I, $I = 1, \ldots, k$, are some Casimir functions with $L = (C^I)^{-1}(0)$ and α_I are closed 1-forms on Σ. For fixed $\widetilde{\Omega}$ and C^I, redefining α_I by adding an exact 1-form on Σ is a gauge transformation. For fixed map $\mathcal{X}$, gauge equivalence classes of solutions to the field equations for A correspond to the set of k elements $[\alpha_I] \in H^1(\Sigma)$.

Here, $H^1(\Sigma)$ denotes the first cohomology of Σ over the real numbers. Eq. (4.1) demonstrates that, in contrast to the moduli space of flat connections (see section 2.2.1), the A-solutions of Poisson Sigma Models corresponding to X-solutions lying in regular leaves are not classified by the fundamental group $\pi_1(\Sigma)$ but by its abelianization, which is $H^1(\Sigma)$. This point will be clarified in the examples below when we will reexamine the case of two-dimensional gauge theories from the Poisson Sigma Model point of view.
Proof: Let A be a solution to the equations (2.1) and (2.2) and $\widetilde{\Omega}$ be a presymplectic form compatible with $\mathcal{P}$. We first introduce coordinates (X^α, X^I) on the neighborhood U of L in M adapted to the foliation such that the X^α coordinatize a leaf L and the X^I are transversal, and show that in these coordinates $\underline{i} \in (\alpha, I)$

$$\mathrm{d}(A_{\underline{i}} + \mathcal{X}^*(\partial_{\underline{i}} \lrcorner \widetilde{\Omega})) = 0\,. \tag{4.2}$$

Using that $\widetilde{\Omega}$ is compatible with $\mathcal{P}$, eq. (2.1) immediately implies

$$A_\alpha = -\widetilde{\Omega}_{\alpha\beta} \mathrm{d}X^\beta = -\mathcal{X}^*(\partial_\alpha \lrcorner \widetilde{\Omega}) \tag{4.3}$$

which shows (4.2) for tangential components.

For transversal components, eq. (2.2) with the expression for A_α leads to

$$\mathrm{d}A_I + \mathcal{P}^{\alpha\beta}{}_{,I}\, \widetilde{\Omega}_{\gamma\alpha} \widetilde{\Omega}_{\delta\beta} \mathrm{d}X^\gamma \mathrm{d}X^\delta = 0$$

where tangential components of the matrix $-\widetilde{\Omega}\mathcal{P}_{,I}\,\widetilde{\Omega}$ appear. Owing to compatibility of $\widetilde{\Omega}$, which for tangential components implies inverseness $(\mathcal{P}\widetilde{\Omega})^\alpha_\beta = \delta^\alpha_\beta = (\widetilde{\Omega}\mathcal{P})^\alpha_\beta$, as well as adaptedness of the coordinates, which implies $\mathcal{P}^{\alpha I} = 0 = \mathcal{P}^{IJ}$, the tangential components fulfill the equation $(\widetilde{\Omega}\mathcal{P}\widetilde{\Omega})_{\alpha\beta} = \widetilde{\Omega}_{\alpha\beta}$. Taking a derivative with respect to X^I yields $-(\widetilde{\Omega}\mathcal{P}_{,I}\,\widetilde{\Omega})_{\alpha\beta} = \widetilde{\Omega}_{\alpha\beta,I}$ and thus

$$\mathrm{d}A_I + \widetilde{\Omega}_{\gamma\delta,I}\mathrm{d}X^\gamma \mathrm{d}X^\delta = 0\,.$$

Note that when we take the derivative, we need $\widetilde{\Omega}$ to be compatible with $\mathcal{P}$ in a whole neighborhood of the leaf and not just on the leaf itself. Using that the second term is nothing but the Lie derivative of $\widetilde{\Omega}$ with respect to ∂_I and that $\widetilde{\Omega}$ is closed, we can reexpress this term as $\mathrm{d}\partial_I\lrcorner\widetilde{\Omega}$. This proves eq. (4.2) for all adapted coordinates.

It now follows directly from (4.2) that

$$A_{\underline{i}} = -\mathcal{X}^*(\partial_{\underline{i}}\lrcorner\widetilde{\Omega}) + \alpha_{\underline{i}}$$

for a set of closed 1-forms $\alpha_{\underline{i}}$ on Σ. According to eq. (4.3) these 1-forms have to vanish for components of A tangential to L. This then establishes eq. (4.1) as a necessary condition for the solutions $A_{\underline{i}}$ and also for A_i in arbitrary coordinates since eq. (4.1) as well as eqs. (2.1), (2.2) are target-space covariant.

Sufficiency follows from the equivalence of eqs. (2.1), (2.2) with (4.3), (4.2) and the restriction on $X(x)$ found in theorem 1.

Eq. (4.1) is already covariant with respect to the gauge transformations of theorem 1 (a change of $X^\alpha(x)$ induces the corresponding change of A_α according to this equation, which reduces to eq. (4.3)). As seen best in adapted coordinates, independently of those transformations, the gauge transformations (2.5) allow us to change the transversal components A_I by adding exact 1-forms (analogously to the discussion in section 3.1). Thus, for a fixed map $\mathcal{X}$, gauge equivalence classes of solutions are given by the cohomology classes $H^1(\Sigma)$ of α_I. This demonstrates the last assertion of the theorem. □

In the proposition, the Casimir functions C^I and the compatible presymplectic form $\widetilde{\Omega}$ were assumed to be fixed. Any other set C_2^I of Casimir functions can be obtained by a map $C_2^I = f_J^I(C)C^J$ where f_J^I are differentiable functions of the original Casimir functions forming an invertible matrix. We then have

$$\mathcal{X}^*(\partial_i C_2^I) = \mathcal{X}^*\left(f_J^I\partial_i C^J + C^J\partial_i f_J^I\right) = X^*\left(\left(f_J^I + C^K\partial_J f_K^I\right)\partial_i C^J\right) = f_J^I(0)\mathcal{X}^*\left(\partial_i C^J\right)$$

and the 1-forms α'_I corresponding to the Casimir functions C_2^I are given by

$$\alpha'_I = (f^{-1})_I^J(0)\alpha_J\,. \tag{4.4}$$

This is just a linear recombination of the original 1-forms, but it can still change the classes in $H^1(\Sigma)$.

Choosing a different compatible presymplectic form $\widetilde{\Omega}$ also implies a redefinition of the 1-forms α_I. As remarked after lemma 1, the freedom in $\widetilde{\Omega}$ is given by adding a closed

2-form λ which vanishes when pulled back to a leaf, i.e. $\mathrm{d}\lambda = 0 = \iota_L^* \lambda$. In a neighborhood of the leaf, such a 2-form can always be written as $\lambda = \beta_I \wedge \mathrm{d}C^I + \gamma_{IJ} \mathrm{d}C^I \wedge \mathrm{d}C^J$ with 1-forms β_I and functions γ_{IJ} which fulfill $\mathrm{d}\beta_I = -\partial_K \gamma_{IJ} \mathrm{d}C^J \wedge \mathrm{d}C^K$. The last equation implies $\mathrm{d}\mathcal{X}^* \beta_I = \mathcal{X}^* \mathrm{d}\beta_I = 0$ since the image of $\mathcal{X}$ lies in the leaf L where the C^I are constant. If we change the compatible presymplectic form to be $\widetilde{\Omega}_2 = \widetilde{\Omega} + \lambda$, we obtain

$$
\begin{aligned}
-\mathcal{X}^*(\partial_i \lrcorner \widetilde{\Omega}_2) &= -\mathcal{X}^*(\partial_i \lrcorner \widetilde{\Omega}) - \mathcal{X}^*(\partial_i \lrcorner \lambda) \\
&= -\mathcal{X}^*(\partial_i \lrcorner \widetilde{\Omega}) - \mathcal{X}^*(\beta_I(\partial_i) \wedge \mathrm{d}C^I - \partial_i C^I \beta_I + 2\partial_i C^I \gamma_{IJ} \mathrm{d}C^J) \\
&= -\mathcal{X}^*(\partial_i \widetilde{\Omega}) + \mathcal{X}^*(\beta_I \partial_i C^I)
\end{aligned}
$$

and the 1-forms are changed to

$$\alpha_I' = \alpha_I + \mathcal{X}^* \beta_I \,. \tag{4.5}$$

As shown above, $\mathcal{X}^* \beta_I$ is closed so that the new α_I' still define elements of the first cohomology. However, the $\mathcal{X}^* \beta_I$ need not be exact and so also a redefinition of $\widetilde{\Omega}$ may change the cohomology classes characterising a given connection (not just the representatives of the original classes).

This implies that there is no *canonical* isomorphism between gauge equivalence classes of A-fields for fixed $\mathcal{X}$ with the set of k elements in $H^1(\Sigma)$, while still any particular choice of $\widetilde{\Omega}$ and C^I does define an isomorphism. The situation is comparable to that in non-abelian gauge theories where a map between gauge equivalence classes of gauge fields and elements of $\pi^1(\Sigma)$ is defined only upon choosing closed curves which generate $\pi^1(\Sigma)$.

4.3.2 Generalizations

So far we assumed that the leaf L has trivial holonomy, implying that there is a global set of Casimir functions C^I such that $L = (C^I)^{-1}(0)$. In other words, the conormal bundle $(NL)^* = \bigcup_{X \in L} \{\alpha \in T_X^* M : \alpha(v) = 0 \text{ for all } v \in T_X L\}$ of L is a trivial vector bundle with global basis $\{\mathrm{d}C^I\}_{I=1\ldots k}$. As seen in Prop. 2, for a fixed leaf L solutions A to the Poisson Sigma Model are then given in terms of closed 1-forms $\alpha = \alpha_I \, \text{đ}C^I$ on Σ, taking values in the pull back of the conormal bundle of L. Here we denoted the basis in the pull back bundle which corresponds to $\mathrm{d}C^I$ by $\text{đ}C^I$; it is not to be confused with the pull back of $\mathrm{d}C^I$, which would be identically zero (and also a section in a different bundle). The respective contribution to (4.1) then can be understood in the following way: Take α as a section in $T^*\Sigma \otimes \mathcal{X}^*(NL)^*$, i.e. $\alpha \in \Omega^1(\Sigma, \mathcal{X}^*(NL)^*)$. $\mathcal{X}^*(NL)^*$ is embedded canonically into $\mathcal{X}^* T^* M$, we thus may view α also as a particular section through that bundle. Then we can contract α with ∂_i, viewed as a basis in $\mathcal{X}^* TM$. Thus the second part of (4.1) can be written as $\langle \partial_i, \alpha \rangle$. Up to gauge transformations, only equivalence classes $[\alpha] \in H^1(\Sigma, \mathcal{X}^*(NL)^*)$ are representatives. This will be made more precise in the more general setting to follow.

If the leaf L has non-trivial holonomy, its conormal bundle is non-trivial and (4.1) cannot be used as a global expression. Instead, we have to choose a covering of the leaf such that in any neighborhood $\mathcal{U}_i$ of the covering there exist Casimir functions $C_{(i)}^I$ specifying $L \cap \mathcal{U}_i = (C_{(i)}^I)^{-1}(0)$. If two neighborhoods have non-empty intersection, there are different sets of Casimir functions which are related by a transformation $C_{(i)}^I = f_{(ij)}{}^I{}_J(C_{(j)}) C_{(j)}^J$ as discussed in the previous subsection.

Example. Let $M = [-1,1]^3/\sim$ where the identification $\sim$ is defined by $(1,y,z) \sim (-1,-y,z)$ for all $y,z \in [-1,1]$, equipped with the Poisson tensor $\mathcal{P} = \partial_x \wedge \partial_z$ admitting the compatible presymplectic form $\widetilde{\Omega} = \mathrm{d}x \wedge \mathrm{d}z$. Any section $z = \mathrm{const}$ is a Möbius strip. The set $L\colon y = 0$ is a leaf in $(M,\mathcal{P})$ (while a set $y = \mathrm{const} \neq 0$ is only half of a leaf) which can be covered by two neighborhoods $\mathcal{U}_{1,2}$ admitting the local Casimir function $C = y$. On the full leaf, however, y is not a global Casimir function since values $y = c$ and $y = -c$ belong to the same leaf for any constant $c \in [-1,1]\backslash\{0\}$. Correspondingly, we have non-trivial transition functions $f_1 = 1$ and $f_2 = -1$ in the two intersections of the neighborhoods.

In any neighborhood, (4.1) is the local expression of a solution with $(\alpha_I\, \text{đ}C^I)_{(i)}$ representing the solution as local section of the conormal bundle (plus the cotangent bundle of Σ certainly). The transformation between different charts is done via (4.4) such that $(\alpha'_I\, \text{đ}C'^I)_{(i)} = (\alpha_I\, \text{đ}C^I)_{(j)}$ in $\mathcal{U}_i \cap \mathcal{U}_j$. As a global object, therefore, the local sections $(\alpha_I\, \text{đ}C^I)_{(i)}$ form again a 1-form α on Σ taking values in the pull back of the conormal bundle of L.

Every local section has to be closed according to the field equations, and they combine to a global 1-form which is closed in the following sense: The transition functions of a non-trivial conormal bundle are given by the *constants* ${f_{(ij)}}^I{}_J$, implying that the conormal bundle is a flat vector bundle[6] with a canonical derivative operator D. To be more explicit, one may define an operator D which annihilates any local basis $(\text{đ}C^I)_{(i)}$; this derivative is then extended to forms on Σ with values in $\mathcal{X}^*(NL)^*$ by the graded Leibniz rule: For $\beta \in \Omega^p(\Sigma, \mathcal{X}^*(NL)^*)$ locally we have $\beta = (\beta_I \text{đ}C^I)_{(i)}$ and then simply $D\beta = (\mathrm{d}\beta_I \text{đ}C^I)_{(i)}$. The locally d-closed 1-forms $(\alpha_I\, \text{đ}C^I)_{(i)}$ combine to a globally D-closed section $\alpha \in \Omega^1(\Sigma, \mathcal{X}^*(NL)^*)$, which represents a solution to the field equations. Clearly, $D^2 = 0$ (since the connection is flat, by construction) and there is a natural cohomology defined on $\Omega^p(\Sigma, \mathcal{X}^*(NL)^*)$, denoted by $H^p(\Sigma, \mathcal{X}^*(NL)^*)$.

To find unique representatives, we have to consider the symmetry transformations. As before, the local expressions $(\alpha_I\, \text{đ}C^I)_{(i)}$ can be changed by adding an exact local 1-form $(\mathrm{d}\epsilon_I \text{đ}C^I)_{(i)}$. Again, the local 1-forms combine to a D-exact 1-form $D(\epsilon_I \text{đ}C^I)$ taking values in the pulled back conormal bundle. Thus, solutions are classified by the first D-cohomology $H^1(\Sigma, \mathcal{X}^*(NL)^*)$.

A similar strategy can be used to deal with leaves which do not admit a compatible presymplectic form: those leaves can be cut into parts each admitting a compatible presymplectic form which have to be glued together by the transformation of the preceding subsection. This will then imply a transformation (4.5) for the α_I on overlapping charts. Returning back to already existing charts then leads to restrictions on the permitted αs. This may lead also to compactifcations in the solution space, which in previous cases was always non-compact while it would be compact for, e.g., 2d BF-theories with compact gauge groups. In the present paper, however, we do not intend to work this out in more detail.

[6]We are grateful to A. Kotov for pointing this out to us.

4.3.3 Almost topological models

Adding a term (1.3) to the action changes the field equation for A, so we will also obtain different solutions. However, the changes are not too drastic. To show this we consider the slightly more general case of an additional term

$$\int_\Sigma C^\sigma(X(x))\,\varepsilon_\sigma \tag{4.6}$$

where a sum of such terms with possibly different Casimir functions C^σ appears. The contribution of this addition to the field equations has been determined in (2.3). Using a set of k functionally independent Casimir functions C^I, all the C^σ can be expressed in terms of these functions (at least locally). Then $C^\sigma{}_{,i} = C^\sigma{}_{,I}\; C^I{}_{,i}$; introducing $\varepsilon_I := \varepsilon_\sigma C^\sigma{}_{,I}$ (a 2-form with values in the conormal bundle; we will make use of this observation below), using (2.3), the key equation (4.2) in the preceding proof is changed to

$$\mathrm{d}(A_{\underline{i}} + \mathcal{X}^*(\partial_{\underline{i}} \lrcorner \widetilde{\Omega})) = -\varepsilon_I \mathcal{X}^*(\partial_{\underline{i}} C^I) \tag{4.7}$$

while eq. (4.3) still holds true due to $\partial_\alpha C^I = 0$. Let us first assume that $\varepsilon_\sigma = \mathrm{d}p_\sigma$ (if some ϵ are taken to be a volume form as in Yang-Mills theories, this can be the case only if Σ is non-compact); then also $\varepsilon_I = \mathrm{d}p_I$ is exact (since $C^\sigma{}_{,I}$ in $p_I = p_\sigma C^\sigma{}_{,I}$ is, as a function of Casimirs, constant on Σ). With the assumption, any solution A can still be cast into the form (4.1), but now the α_I are not necessarily closed but only $\alpha_I + p_I$ (by assumption not all p_I can be closed, since otherwise all $\varepsilon_I = \mathrm{d}p_I$ would vanish). Noting that the symmetries are unaltered, we obtain

Corollary 3. *For a given map $\mathcal{X}$ with image in a symplectic leaf L of trivial holonomy which has a neighborhood U permitting a presymplectic form $\widetilde{\Omega}$ compatible with $\mathcal{P}$, any solution to the field equations (2.1), (2.3) with $\varepsilon_I = \mathrm{d}p_I$ may be written in the form (4.1) where C^I, $I = 1, \ldots, k$, are some Casimir functions with $L = (C^I)^{-1}(0)$. The 1-forms $\alpha_I + p_I$ are closed on Σ. For fixed $\widetilde{\Omega}$ and C^I, redefining α_I by adding an exact 1-form on Σ is a gauge transformation. For a fixed map $\mathcal{X}$, gauge equivalence classes of solutions to the field equations for A correspond to the set of k elements $[\alpha_I + p_I] \in H^1(\Sigma)$.*

If an ε_I is not exact, we have to proceed more carefully. For an exact $\varepsilon_I = \mathrm{d}p_I$ we have just seen that the α_I have to fulfill $\mathrm{d}\alpha_I = -\mathrm{d}p_I = -\varepsilon_I$. Locally, this will still hold for a non-exact ε_I as a consequence of the field equations, but there will be no 1-forms α_I which can fulfill this equation globally.

For simplicity we first discuss the case $k = 1$, i.e. that there is only one α which locally fulfills $\mathrm{d}\alpha = \epsilon$. If we choose a good cover $\{\mathcal{U}_m\}$ of Σ (the neighborhoods $\mathcal{U}_m$ as well as their nonvanishing intersections are topologically trivial), then in each $\mathcal{U}_m$ we have a 1-form p_m such that $\mathrm{d}p_m = \epsilon$. Furthermore, due to $\mathrm{d}(p_m - p_n) = \epsilon - \epsilon = 0$, we have functions λ_{mn} with $p_m - p_n = \mathrm{d}\lambda_{mn}$ on $\mathcal{U}_m \cap \mathcal{U}_n$. Now, α has to fulfill $\mathrm{d}\alpha_m = \epsilon = \mathrm{d}p_m$ which implies $\alpha_m = p_m + \mathrm{d}\lambda_m$ for some functions λ_m which can be chosen to be zero by redefinition of p_m or by using an appropriate local gauge transformation. In the intersection of two neighborhoods $\mathcal{U}_m$ and $\mathcal{U}_n$, the local 1-forms of α do not necessarily agree but differ by an exact form: $\alpha_m - \alpha_n = \mathrm{d}\lambda_{mn}$. In other words, α is a connection on the line bundle with curvature ϵ which is obtained as the pull back of the conormal bundle of the leaf L. As usually, we have $(\alpha-\alpha')_m - (\alpha-\alpha')_n = 0$ such that the difference of two such connections α and α' is a global 1-form. It has to fulfill $\mathrm{d}(\alpha-\alpha') = 0$, i.e., it is closed. Therefore, the space of all connections of curvature ϵ can be identified with the space of closed 1-forms, and the space of gauge equivalence classes with the first cohomology $H^1(\Sigma)$.

If $k > 1$, we obtain the α_I as k connections on k line bundles the I-th one of which has curvature ε_I. Alternatively, the α_I together can be viewed as components of an $\mathfrak{a}$-connection on the pull back of the conormal bundle of the leaf L, where $\mathfrak{a}$ is the transversal Lie algebra of the leaf. In fact, the part $\alpha := \alpha_I C^I{}_{,i}\, \text{đ}X^i$ of (4.1), $\text{đ}X^i$ denoting a basis in $\mathcal{X}^* T^*_X M$, in a point $X \in L$ always takes values in the transversal Lie algebra $\mathfrak{a}_X$ which, as a manifold, can be identified with the conormal space $(N_X L)^*$. Furthermore, the transversal Lie algebra of a regular leaf is always abelian and, therefore, isomorphic to $\mathbb{R}^k$ with $k = \mathrm{codim}(L, M)$ which coincides with our result of k abelian connections on k line bundles. For a regular leaf the reformulation via the transversal Lie algebra is thus almost trivial, but we will see later that it is helpful for a possible generalization to non-regular leaves.

Finally, gauge transformations of the Poisson Sigma Model have already been seen to add exact 1-forms to α_I, which agrees with the notion of gauge transformation for a connection. Together with the known classification of inequivalent bundles with connection we obtain

Proposition 3. *For a given map $\mathcal{X}$ with image in a symplectic leaf L of trivial holonomy which has a neighborhood U permitting a presymplectic form $\widetilde{\Omega}$ compatible with $\mathcal{P}$, any solution to the field equations (2.1), (2.3) may be written in the form (4.1) where C^I, $I = 1, \ldots, k$, are some Casimir functions with $L = (C^I)^{-1}(0)$.*

The α_I form a transversal Lie algebra valued connection on the pull back of the conormal bundle of L to Σ with curvature ε_I. For a fixed map $\mathcal{X}$, gauge equivalence classes of solutions to the field equations for A correspond to inequivalent connections of the given curvature on the given line bundle on Σ. All those connections are classified by k elements of $H^1(\Sigma)$.

This agrees with the previous results in the case of vanishing or exact ε_I, in which case the bundles with connection over Σ are trivial. If L has non-trivial holonomy, we can combine Prop. 3 with the result of the previous subsection. On each chart $\mathcal{U}_m$ there is a 1-form α_m with values in the pull back of the conormal bundle (restricted to the chart). Globally there is some 2-form $\varepsilon = \varepsilon_I \text{đ}C^I$, taking values in $\mathcal{X}^*(NL)^*$, furthermore; on local charts $\{\mathcal{U}_m\}$, it has primitives p_m, i.e. $\varepsilon = Dp_m$. Similarly to before we find $D\alpha_m = Dp_m$, concluding $\alpha_m = p_m + D\lambda_m$. With $D\lambda_{mn} = p_m - p_n$ we then obtain on intersections

$$\alpha_n = \alpha_m + D\left(\lambda_{mn} + \lambda_m - \lambda_n\right) , \tag{4.8}$$

where λ_m reflects the ambiguity in the definition of the local primitives p_m or likewise the local gauge freedom. This defines a kind of connection on the conormal bundle with

JHEP07(2003)002

"curvature" $\varepsilon = \varepsilon_I \text{đ}C^I$ (on each chart we have $\varepsilon_m \equiv \varepsilon|_{\mathcal{U}_m} = D\alpha_m$). It would be interesting to clarify the precise mathematical nature of such an object $\{\alpha_m\}$. Since the λ_m can be gauged to zero and the λ_{mn} are fixed by the 2-forms ε_I (up to the previously mentioned ambiguity given by λ_m), the *difference* between two such collections $\{\alpha_m\}$ again defines a global 1-form with values in $\mathcal{X}^*(NL)^*$. Thus the space of all inequivalent α of the given "curvature" ε is classified by the first cohomology of conormal bundle valued forms.

Despite the fact that the addition of (1.3) spoils the topological nature of the model, corollary 1 and proposition 3 show that the moduli space of classical solutions is parameterized by the same topological objects which classify solutions of the topological models.

4.3.4 Summary

Let us summarize the results of this section in

Theorem 2. *Let Σ be a two-dimensional manifold and $(M, \mathcal{P})$ a Poisson manifold.*

For stationary points of a topological or almost topological Poisson Sigma Model with Σ and $(M, \mathcal{P})$ the image of the map $\mathcal{X} \colon \Sigma \to M$ is contained in a symplectic leaf L of M. If L admits a compatible presymplectic form, the space of corresponding solutions for A is given by the first cohomology class $H^1(\Sigma, \mathcal{X}^*(NL)^*)$ of forms on Σ taking values in the pull back via $\mathcal{X}$ of the conormal bundle of the leaf L. A local representation of A-solutions is given by (4.1).

We already remarked on a possible generalization to cases where a compatible presymplectic form does not exist globally (section 4.3.2). Later we will in particular discuss the case of non-regular leaves.

5. Examples

Applying theorem 2, we see that in all cases where all leaves have the same codimension k the solution space $\mathcal{M}_{\text{cl}}(\Sigma)$ is of dimension

$$\dim \mathcal{M}_{\text{cl}}(\Sigma) = k \left(\operatorname{rank} H^1(\Sigma) + 1\right)$$

which generalizes formula (3.2) for the dimension in the topologically trivial case. However, in the general case the solution space will not be a linear space because there may be non-trivial identifications, which depend on the topology of Σ and the leaf L and can even lead to a non-Hausdorff topology by gluing the sectors corresponding to different homotopy classes of maps $\mathcal{X} \colon \Sigma \to L$. Specializing theorem 2 to the topologically trivial case dealt with in section 3.1 shows that we get back the explicit solutions given there. But theorem 2 is applicable to a class of Poisson Sigma Models more general by far. E.g. if M is foliated trivially but by topologically non-trivial leaves provided only their second cohomology vanishes, all solutions are given by theorem 2 owing to lemma 1.

We can also compare with the results obtained in section 3.2 for non-abelian BF-theories. In this case M is the Lie algebra $\mathfrak{g}$ of a semisimple Lie group G equipped with the Poisson tensor $\mathcal{P}^{ij} = f^{ij}{}_k X^k$ and the symplectic leaves are identical to the adjoint orbits in $\mathfrak{g}$ (we identify the Lie algebra of a semisimple Lie group with its dual by means

JHEP07(2003)002

of the Cartan-Killing metric). As compared to section 3.2, we are now solving the field equations in the opposite direction, i.e. we first solve for X. According to theorem 1, all equivalence classes of solutions are given by homotopy classes of maps $\mathcal{X}\colon \Sigma \to L$ for any leaf L. This is identical to the results found in section 3.2. Now, given a solution X, solutions for A are given by theorem 2 in those cases in which a compatible presymplectic form exists. This can be the case only for non-compact G: for compact G all leaves in M are compact symplectic manifolds which necessarily have non-trivial second homology, for otherwise their symplectic form would be exact and so the symplectic volume would vanish. Now appealing to corollary 2 shows that there is no compatible presymplectic form in any neighborhood of a given regular leaf because the symplectic volume of any non-trivial two-cycle is not constant along the direction ∂_C given by the Casimir function $C(X) = \mathrm{tr}(X^2)$. Recall that for compact G the solution space, i.e. the space of flat connections, is compact, which also demonstrates that in this case our methods cannot be applicable (theorem 2 always implies a non-compact solution space). As discussed at the end of section 4.3.2, solutions for leaves which do not admit a global presymplectic form can be found by gluing solutions obtained with local forms. The gluing procedure will lead to additional identifications which can compactify the solution space.

If there is a compatible presymplectic form for a non-compact group G (if, e.g., all leaves have trivial second cohomology, cf. lemma 1), we can apply theorems 1 and 2 in order to find solutions. Solutions for X are given by maps $\mathcal{X}\colon \Sigma \to M$ with image contained in an adjoint orbit which coincides with the observations in section 3.2. However, general results about the existence of compatible presymplectic forms are available only for a non-degenerate leaf such that theorem 2 can directly only lead to solutions with reducible connections for a semisimple group G. In fact, in simple cases one can show easily that a connection of the form (4.1) is reducible: If $\mathrm{rank}\, G = 1$ the Casimir function is $C = \mathrm{tr}(X^2)$, denoting the Cartan-Killing norm on $\mathfrak{g}$ by tr, and (4.1) takes the form (using generators T^i of G)

$$A_i T^i = -T^i \mathcal{X}^*(\partial_i \lrcorner \widetilde{\Omega}) + 2\alpha X$$

with a closed 1-form α on Σ. If $\mathcal{X}$ can be gauged to be a constant map, the first term vanishes leading to $A = 2\alpha X$. This implies that all holonomies of A are given by $\exp cX$ for some $c \in \mathbb{R}$ which shows that A is reducible. We can, therefore, expect to have access to the generic part of the solution space only if $\mathrm{rank}\, \pi_1(\Sigma)$ is small (see, however, possible generalizations discussed in the next section). Otherwise, the solution space would be dominated by irreducible connections which lead to X-solutions in the degenerate leaf given by the origin. For $\mathrm{rank}\, \pi_1(\Sigma) \leq 1$, which physically is most interesting, theorem 2 determines the generic part of $\mathcal{M}_{\mathrm{cl}}(\Sigma)$ because there are no irreducible connections. In fact, the dimensions of the solution spaces given in section 3.2 and theorem 2 coincide: in both cases we need $k = \dim \ker \mathcal{P}$ parameters to specify a leaf, which in turn determines the equivalence class of an X-solution (up to certain discrete labels which we need in order to fix the homotopy class of the map $\mathcal{X}$), and $k\, \mathrm{rank}\, H^1(\Sigma)$ parameters to specify the A-solution. Note that $k = \dim \ker \mathcal{P} = \dim G/\mathrm{Ad}$ so that the dimensions of the space of reducible flat connections and of the solution space according to theorem 2 in fact coincide.

Noting that, as already remarked in section 3.2, X-solutions lying in regular leaves correspond to reducible connections whose holonomies generate a maximal *abelian* subgroup of G, we can clarify the appearance of $H^1(\Sigma)$ in the Poisson Sigma Model classification of A-solutions as opposed to $\pi_1(\Sigma)$ in the gauge theory classification: Since all holonomies commute, only the abelianization of $\pi_1(\Sigma)$ matters, which is just $H^1(\Sigma)$.

Corollary 1 and proposition 3 in particular provide solutions for Yang-Mills theories when we choose a quadratic Casimir C and volume form ϵ. Proposition 3 only applies if the X-solution maps Σ into a non-degenerate leaf, so that we obtain solutions with non-vanishing electric field X leading to a non-flat connection. The difference between BF- and Yang-Mills theories is automatically accounted for by the appearance of ϵ in the conditions for a solution A.

6. Non-regular leaves

Since we are not aware of general results concerning the existence of compatible presymplectic forms for non-regular leaves, theorem 2 does not give us direct access to solutions in this case. The comparison with non-abelian BF-theories shows that in general we cannot expect non-regular leaves to contribute only a lower-dimensional set to $\mathcal{M}_{\rm cl}(\Sigma)$; in fact those leaves usually correspond to solutions forming a dense subset of the moduli space. Only if the rank of the fundamental group of Σ does not exceed one is the moduli space dominated by solutions corresponding to non-degenerate leaves. As we will see below, this holds true also for non-linear Poisson structures.

But the information we obtain is of interest also in cases where solutions for regular leaves do not correspond to the generic part of $\mathcal{M}_{\rm cl}(\Sigma)$ in a given model and complements methods which are targeted to the generic part (e.g. the theory of irreducible flat connections on compact Riemann surfaces used in two-dimensional non-abelian gauge theories).

The case of gravitational models is special because we have an additional condition which requires the metric constructed from A to be non-degenerate. Investigations with other methods [21, 22] suggest that this reduces the contributions from non-regular leaves such that the methods developed here can have access to the main part of the moduli space. In the present paper, however, we will not dicuss this issue further and instead focus on a possible generalization of the classification of solutions to non-regular leaves.

When discussing the solutions for almost topological Poisson Sigma Models we already observed that the connection has to be transversal Lie algebra valued, which in the case of regular leaves is always an abelian algebra. To generalize this result we first recall how the transversal Lie algebra of a point $X \in L$ of a leaf L can be constructed [35]:

Definition 1. *The* transversal Lie algebra *of a point X in a leaf L of a Poisson manifold $(M, \mathcal{P})$ is the conormal space $\mathfrak{a}_X := (NL_X)^* = (T_XM/T_XL)^* \equiv (T_XL)^0 = \{\alpha \in T^*_XM : \alpha(v) = 0$ for all $v \in T_XL\}$, identified with the annihilator of the tangent space T_XL, with the following Lie bracket: For two elements $\alpha, \beta \in \mathfrak{a}_X$ we choose functions f and g which vanish in a neighborhood of X in the leaf L such that $\mathrm{d}f_X = \alpha$ and $\mathrm{d}g_X = \beta$. The bracket*

$$[\alpha, \beta]_X := \mathrm{d}\{f, g\}_X \tag{6.1}$$

is then well defined and defines the transversal Lie algebra $\mathfrak{a}_X$.

We will later use another way to identify $\mathfrak{a}_X$ as a submanifold of the cotangent bundle of M:

Lemma 2. *As a manifold, the transversal Lie algebra $\mathfrak{a}_X$ is the kernel of the Poisson tensor $\mathcal{P}$ in X.*

Proof: For any cotangent vector $\omega \in T^*_X M$ the vector $v := \mathcal{P}^{\#}(\omega)$ is tangential to L such that $\mathcal{P}(\alpha, \omega) = \alpha(v) = 0$ for all $\omega \in T^*_X M$ proving that $\mathfrak{a}_X$ is contained in the kernel of $\mathcal{P}$. Equality of the vector spaces then follows from a dimensional argument. □

We are going to discuss the transversal Lie algebra for gauge theories where $M = \mathfrak{g}^*$ as in section 2.2.1.

Lemma 3. *If $M = \mathfrak{g}^*$ is the dual of a Lie algebra, then the transversal Lie algebra $\mathfrak{a}_X$ of a point $X \in M$ is the isotropy algebra of the co-adjoint action of $\mathfrak{g}$ at X.*

Proof: As a subspace of the cotangent bundle of $\mathfrak{g}^*$, $\mathfrak{a}_X$ is naturally identified with a subspace of $\mathfrak{g}^{**} \equiv \mathfrak{g}$. Furthermore, it follows from the definition that $\mathfrak{a}_X$ is also a subalgebra of $\mathfrak{g}$: if $\alpha = \mathrm{d}f_X = f_{,i}\, \mathrm{d}X^i$ and $\beta = \mathrm{d}g_X = g_{,i}\, \mathrm{d}X^i$ are in the kernel of $\mathcal{P}^{\sharp}_X$, we have

$$\mathrm{d}\{f, g\} = \mathrm{d}(f^{ij}{}_k X^k f_{,i}\, g_{,j}\,) = f^{ij}{}_k f_{,i}\, g_{,j}\, \mathrm{d}X^k$$

which implies

$$[\alpha, \beta]_X = \mathrm{d}\{f, g\} = f^{ij}{}_k \alpha_i \beta_j \mathrm{d}X^k \equiv [\alpha, \beta]_{\mathfrak{g}}$$

where the last bracket denotes the usual bracket in the Lie algebra $\mathfrak{g}$ and $\mathrm{d}X^k$ are identified with the generators of $\mathfrak{g}^{**} \equiv \mathfrak{g}$.

The condition for $\mathfrak{a}_X$ of lemma 2 now reads

$$\mathcal{P}_X(\alpha, \omega) = X([\alpha, \omega]_{\mathfrak{g}}) = (\mathrm{coad}_\alpha X)(\omega) = 0$$

for all $\omega \in \mathfrak{g}$ which concludes the proof. □

If $\mathfrak{g}$ is semisimple, we can identify $M = \mathfrak{g}^*$ with $\mathfrak{g}$ and the point $X \in M$ with an element of $\mathfrak{g}$. The transversal Lie algebra $\mathfrak{a}_X$ then is the subalgebra of $\mathfrak{g}$ fulfilling $[X, \alpha] = 0$ for all $\alpha \in \mathfrak{a}_X$.

We are now ready to exploit this information in the context of solutions to gauge theories. We already know that solutions for the field X are given by arbitrary maps of Σ into a leaf L of M. Locally, the map $\mathcal{X}$ can be deformed by gauge transformations such that its image is a single point $X \in L$; therefore, the field equation $\mathrm{d}X + [A, X] = 0$ implies that all solutions for A have to commute with $X \in \mathfrak{g}$ and thus, according to lemma 3, have values in the transversal Lie algebra of X. Furthermore, the local component 1-forms A_i such that $A = A_i \mathrm{d}X^i$ of all those connections can be written as $\alpha_I C^I{}_{,i}$ with k 1-forms α_I where k is the codimension of the leaf and C^I are k Casimir functions specifying the leaf. If the image of X is not just a single point, we need an additional contribution a for the connection such that $[a, X] = -\mathrm{d}X$. Then, $A := a + \alpha_I C^I{}_{,i} T^i$ would provide a solution to

$\mathrm{d}X + [A, X] = 0$. If we can find such a form a, any solution to the first field equation can be written as a plus a transversal Lie algebra valued connection. This demonstrates that the role of the transversal Lie algebra is unchanged if we have a non-regular leaf. Now it is easy to see that $a = -\partial_i \lrcorner \widetilde{\Omega} T^i$ with $\widetilde{\Omega}$ compatible with $\mathcal{P}$ is appropriate because

$$[\partial_i \lrcorner \widetilde{\Omega} T^i, X] = f^{ij}{}_k \widetilde{\Omega}_{il} \mathrm{d}X^l X_j T^k = \mathrm{d}X^k T_k$$

provided that $f^{ij}{}_k X_j \widetilde{\Omega}_{i\alpha} = -\mathcal{P}^{ik} \widetilde{\Omega}_{i\alpha} = \delta^k_\alpha$ (where α can be regarded as a tangential index since it is contracted with $\mathrm{d}X^l$). Note that for this equation $\widetilde{\Omega}$ only needs to give the leaf symplectic structure when restricted to the leaf itself which is weaker than the condition for a compatible presymplectic form. However, one also has to assure that a has the correct curvature in order for A to solve the second field equation. For regular leaves this requires a to be constructed with a compatible presymplectic form as we have seen.

The last calculations suggest that the classification of solutions to Poisson Sigma Models as found in this paper generalizes to arbitrary leaves where the transversal Lie algebra plays the role of the connection 1-forms α_I. (This is also suggested by a reinterpretation of the solutions as Lie algebroid morphisms [28].) Only the explicit form (4.1) of a solution cannot be used if there is no substitute for the compatible presymplectic form $\widetilde{\Omega}$. One example where one can easily find an alternative form for a non-regular leaf is the origin as a degenerate leaf in a semisimple Lie algebra: Here we can choose $a = 0$ since all maps X into this leaf have only one image point, which would correspond to a form $\widetilde{\Omega}$ which is not compatible with $\mathcal{P}$ in a neighborhood of the leaf. In this case, the transversal Lie algebra agrees with the Lie algebra itself such that all solutions for A are given by Lie algebra valued connections with the correct curvature (zero for BF theories or given by the volume form for Yang-Mills theories). Thus, the methods of the present paper give us the well-known results also for a degenerate leaf, in which case we obtain irreducible connections and a vanishing X.

In fact, this conclusion does not only apply to BF and Yang-Mills, which have a linear Poisson tensor, but to a general Poisson Sigma Model as well provided that the image of $\mathcal{X}$ is contractible in the leaf L. In this case, one can choose the gauge in which the image of $\mathcal{X}$ is a single point where the previous remarks can be used. The A-components tangential to the leaf in the given point must be zero in this gauge owing to the first field equation, while the remaining components are subject to the second field equation with structure *constants* $P^{IJ}{}_{,K}$ of the transversal Lie algebra. Thus, up to gauge transformations, A has to be a flat transversal Lie algebra valued connection whenever the $\mathcal{X} \colon \Sigma \to L$ in (1) is of trivial homotopy — so that X can be gauged to be constant. The remaining gauge freedom then gives the usual gauge transformations of a connection.

In this special case, the field equations of a general Poisson Sigma Model can be reduced to those of BF-theory, and also the formulas we obtained in section 3.2 for the dimensions of subspaces $\mathcal{M}_{\mathrm{cl}}(\Sigma)$ corresponding to different classes of leaves (regular or non-regular) can be used. Regular leaves L_{reg} always contribute solutions which form a subspace of dimension $(\mathrm{rank}\,\pi_1(\Sigma) + 1)\mathrm{codim}(L_{\mathrm{reg}}, M)$ while a degenerate leaf L_{deg} yields $(\mathrm{rank}\,\pi_1(\Sigma) - 2)\mathrm{codim}(L_{\mathrm{deg}}, M)$. Since $\mathrm{codim}(L_{\mathrm{deg}}, M) > \mathrm{codim}(L_{\mathrm{reg}}, M)$ for a Poisson tensor of non-constant rank, the contribution of a degenerate leaf will always dominate

JHEP07(2003)002

the solution space provided that the rank of the fundamental group of Σ is large enough. Similarly, one can see that the dimension of the solution space for any leaf L (not necessarily regular or degenerate) is approximately given by $\operatorname{rank}\pi_1(\Sigma)\operatorname{codim}(L,M)$ for large rank of the fundamental group (this is the contribution of connections taking values in the transversal Lie algebra of dimension $\operatorname{codim}(L,M)$, while the contribution of X-solutions is not proportional to the rank of the fundamental group and thus sub-dominant). Therefore, regular leaves will not give a dense subset of the solution space for large fundamental group of Σ, even when there are no degenerate leaves. In other words, the leaves of the lowest dimension dominate the solution space.

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JHEP07(2003)002

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